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Quantum Zeno effect and light–dark periods for a single atom

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Abstract. The quantum Zeno effect (QZE) predicts a slow-down of the time development of a system under rapidly repeated ideal measurements, and experimentally this was tested for an ensemble of atoms using short laser pulses for non-selective state measurements. Here we consider such pulses for selective measurements on a *single* system. Each probe pulse will cause a burst of fluorescence or no fluorescence. If the probe pulses were strictly ideal measurements, the QZE would predict periods of fluorescence bursts alternating with periods of no fluorescence (light and dark periods) which would become longer and longer with increasing frequency of the measurements. The non-ideal character of the measurements is taken into account by incorporating the laser pulses in the interaction, and this is used to determine the corrections to the ideal case. In the limit, when the time, Δt, between the laser pulses goes to zero, no freezing occurs but instead we show convergence to the familiar macroscopic light and dark periods of the continuously driven Dehmelt system. An experiment of this type should be feasible for a single atom or ion in a trap.

1. Introduction

The effect of an instantaneous measurement on a quantum mechanical system is usually described by the projection postulate of von Neumann and Lüders § according to which, depending on the outcome of a measurement, the wavefunction of the system is projected onto the respective eigenspaces of the observable under consideration. This is also called reduction or collapse of the wavefunction under an ideal measurement; a more general approach to measurements is taken in [4]. Using this concept and some fairly general technical assumptions, Misra and Sudarshan [5] have investigated how a system is affected by rapidly repeated ideal measurements at times Δt apart. They found a slow-down of the system's time development and, in the limit $\Delta t \to 0$, a freezing of the state. This is called the quantum Zeno effect (QZE). The basic reason for this is the fact that for short enough times transition probabilities grow only quadratically with time, not linearly.

To test this effect, Itano *et al* [6] performed an experiment with an ensemble of 5000 ions in a trap (see figure 1 for the relevant level structure, a V configuration). The time development was given by a so-called π pulse of length T_π, tuned to the 1–2 transition

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§ The projection postulate as currently used has been formulated by Lüders [1]. For observables with degenerate eigenvalues his formulation differs from that of von Neumann [2]. It has been pointed out to us by A Sudbury (private communication) that in the first edition of his book Dirac [3] defines observations which cause minimal disturbance and which correspond to Lüder's prescription; in later editions, however, this passage has been omitted.

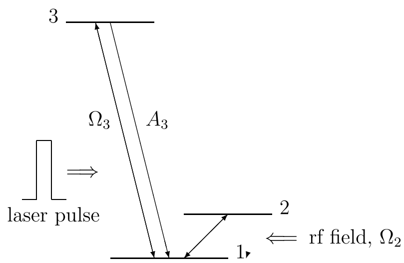

Figure 1. V system with (meta-) stable level two and Einstein coefficient A_3 for level 3. Ω_2 and Ω_3 are the Rabi frequencies of the RF field and the probe laser, respectively.

frequency. A π pulse, here a radio frequency (RF) pulse, transforms the initial state $|1\rangle$ into $|2\rangle$ at the end of the pulse, if no measurements are performed. Following a proposal by Cook [7] the population of the lower level was measured—non-selectively and without actually recording the results—in rapid succession through the fluorescence induced by very short pulses of a strong probe laser which couple level one with an auxiliary third level. The population at time T_π was then measured by a final pulse and recorded. The experimental results were in good agreement with the predictions of the QZE.

The QZE and this experiment have not only aroused considerable interest in the literature [8, 9], but the very relevance of the above experimental results for the QZE has given rise to controversies. In particular the projection postulate and its applicability in this experiment have been cast into doubt, and it was pointed out that the experiment could be understood without recourse to the QZE by simply including the probe laser in the dynamics, e.g. in the Bloch equations or in the Hamiltonian [9]. Since the Bloch equations describe the density matrix of the *complete* ensemble, including the probe pulse as an interaction in them gives, however, no direct insight on how such a pulse acts on a single system.

In previous papers [10–12] we have therefore investigated how far a short laser pulse realizes a selective measurement, i.e. on single systems, to which the projection postulate can be applied. By means of the quantum jump approach (or Monte Carlo wavefunctions or quantum trajectories) [13] and including the probe laser in the dynamics we showed analytically that for a wide range of parameters such a short laser pulse acts, indeed, as an *effective* level measurement to which the usual projection postulate applies with high accuracy. The corrections to the ideal reductions and their accumulation over n pulses were calculated. Our conclusion was that the projection postulate is an excellent pragmatic tool for a quick and intuitive understanding of the slow-down of the time evolution in experiments of this type and that it gives a good physical insight. But it is only approximate, and a more detailed analysis has to take the corrections into account.

The experiment of [6] deals with the effect of repeated non-selective measurements on an ensemble of systems and with the associated slow-down in the time evolution of the density matrix of the total ensemble. It suggests itself to perform a similar experiment with a single atom (or ion) in a trap, though not only for the duration of a π pulse of the weak driving field but instead for an arbitrary *long* time. This might be regarded as an analog of the idealized situation of rapidly repeated measurements on a single system. As studied in [5, 7], in the idealized situation the outcome of the measurements will form a stochastic sequence, in this case a sequence of states $|1\rangle$ and $|2\rangle$. The periods containing only $|1\rangle$'s and $|2\rangle$'s will become increasingly long when the time Δt between the ideal measurements decreases, and in the limit $\Delta t \to 0$ one would have a single infinite sequence of $|1\rangle$'s or $|2\rangle$'s, i.e. freezing. With short pulses of a probe laser, considered as measurements, one would therefore expect periods of fluorescence bursts (light periods, corresponding to periods of $|1\rangle$'s) alternating with periods of no fluorescence (dark periods, corresponding to periods of $|2\rangle$'s). Decreasing the time Δt between the probe pulses should, in this picture, make the light and dark periods longer.

The aim of this paper is to analyse how far this intuitive picture of the behaviour of a single system is correct and to provide an understanding why the projection postulate also works so well in this case. After a brief review of the ideal case we use our previous results to calculate in section 3 the mean duration of the light and dark periods, T_L and T_D, and compare them with the simple expression obtained by the projection postulate. Our analysis will make it perfectly clear why the projection postulate gives such excellent results for a wide range of parameters. If the time Δt between the probe pulses becomes too small, however, then the above simple picture breaks down. In section 4 we will explicitly perform the limit $\Delta t \to 0$ and show that in contrast to the idealized case T_L and T_D remain finite. Indeed, we show convergence to the same expressions as for the famous light and dark periods of the continuously driven Dehmelt system, which are also known under the name of 'electron shelving' [14]. In the last section we discuss our results.

2. Brief review of an ideal case

If one performs rapidly repeated ideal measurements of an observable A with discrete eigenvalues on a single system at times Δt apart then the projection postulate predicts that one will find the same value of A in a row for some time, then another value for some time, and so on. The length of these time intervals is stochastic, and their lengths increase when Δt decreases. For an observable A with non-degenerate discrete eigenvalues this can be seen as follows. For simplicity we make a domain assumption further below. For the general treatment see [5].

Let $|a\rangle$ be a state vector and $\mathbb{P}_a \equiv |a\rangle\langle a|$ the corresponding projector. At times $t_1, t_2, \ldots,$ with $\Delta t \equiv t_{i+1} - t_i$, ideal measurements of $\mathbb{P}_a$ are performed, whose results are 1 or 0, with the system afterwards in $|a\rangle$ or the subspace orthogonal to $|a\rangle$, respectively. This is equivalent to asking whether the result of a measurement is $|a\rangle$ or perpendicular to $|a\rangle$, and we denote the outcome a and $\perp$ instead of 1 and 0. We define $\mathbb{P}_\perp = \mathbb{1} - \mathbb{P}_a$. Let $U(t, t')$ be the time-development operator for the system. If, for initial state $|a\rangle$, one has found a in n successive measurements, the resulting state is, up to normalization, given by

$$|\psi_a(t_n, t_0)\rangle \equiv \mathbb{P}_a U(t_n, t_{n-1})\mathbb{P}_a \ldots \mathbb{P}_a U(t_1, t_0)|\psi\rangle \tag{1}$$

which of course is proportional to $|a\rangle$, and the probability $P_a(t_n, t_0; |\psi\rangle)$ for this is

$$\begin{aligned} P_a(t_n, t_0; |\psi\rangle) &= \| |\psi_a(t_n, t_0)\rangle \|^2 \\ &= |\langle a|U(t_1, t_0)|\psi\rangle|^2 \prod_{i=2}^{n} |\langle a|U(t_i, t_{i-1})|a\rangle|^2. \end{aligned} \tag{2}$$

If one has found $\perp$ in n successive measurements the state is

$$|\psi_\perp(t_n, t_0)\rangle = \mathbb{P}_\perp U(t_n, t_{n-1})\mathbb{P}_\perp \cdots \mathbb{P}_\perp U(t_1, t_0)|\psi\rangle \tag{3}$$

which in general is no longer proportional to a fixed vector, and the probability for this is given by

$$P_\perp(t_n, t_0; |\psi\rangle) = \| |\psi_\perp(t_n, t_0)\rangle \|^2.$$

To show that, for fixed $t = n\Delta t$, $P_a(t, t_0) \to 1|\langle a|\psi\rangle|^2$ for $\Delta t \to 0$ we assume for simplicity that $|a\rangle$ is in the domain of H. An expansion then gives [15]

$$\begin{aligned} |\langle a|U(t_i, t_{i-1})|a\rangle|^2 &= 1 - \Delta t^2[\langle a|HH|a\rangle - \langle a|H|a\rangle^2]/\hbar^2 + \mathrm{o}(\Delta t^2) \\ &= \mathrm{e}^{-\Delta t^2[\langle a|H^2|a\rangle - \langle a|H|a\rangle^2]/\hbar^2}(1 + \mathrm{o}(\Delta t^2)) \end{aligned} \tag{4}$$

where $\mathrm{o}(\Delta t^2)$ denotes terms which go to 0 faster than Δt^2. The expression $\langle a|HH|a\rangle$ is to be interpreted as $||H|a\rangle||^2$. Equation (4) just states the well known fact that under the above assumptions the transition probability from $|a\rangle$ to an orthogonal state goes as Δt^2 for small Δt [16]. From equations (2) and (4) one now obtains for the probability

$$P_a(t, t_0; |\psi\rangle) = \mathrm{e}^{-(n-1)\Delta t^2[\langle a|H^2|a\rangle - \langle a|H|a\rangle^2]/\hbar^2}(1 + \mathrm{o}(\Delta t^2))^{n-1}|\langle a|U(t_1, t_0)|\psi\rangle|^2. \tag{5}$$

With $n = t/\Delta t$ the first and second factor in equation (5) go to 1 for $\Delta t \to 0$, and the last to $|\langle a|\psi\rangle|^2$.

Under the same conditions one can also show that $P_\perp(t, t_0; |\psi\rangle) \to 1\|\mathbb{P}_\perp|\psi\rangle\|^2$ for $\Delta t \to 0$. If $\mathbb{P}_\perp$ were a one- or finite-dimensional projector this would follow as before, but in the general case another argument is needed. With $U_{\Delta t} \equiv U(\Delta t, 0)$ one has from equation (3)

$$\begin{aligned} P_\perp(t_i, t_0; |\psi\rangle) - P_\perp(t_{i+1}, t_0; |\psi\rangle) &= \|\,|\psi_\perp(t_i, t_0)\rangle\|^2 - \|(\mathbb{1} - |a\rangle\langle a|)U_{\Delta t}|\psi_\perp(t_i, t_0)\rangle\|^2 \\ &= \langle a|U_{\Delta t}|\psi_\perp(t_i, t_0)\rangle\langle\psi_\perp(t_i, t_0)|U^*_{\Delta t}|a\rangle. \end{aligned} \tag{6}$$

Using $|\psi_\perp(t_i, t_0)\rangle\langle\psi_\perp(t_i, t_0)| \leqslant \mathbb{1} - |a\rangle\langle a|$ one obtains

$$\begin{aligned} P_\perp(t_i, t_0; |\psi\rangle) - P_\perp(t_{i+1}, t_0; |\psi\rangle) &\leqslant 1 - |\langle a|U_{\Delta t}|a\rangle|^2 \\ &= \Delta t^2[\langle a|HH|a\rangle - \langle a|H|a\rangle^2]/\hbar^2 + \mathrm{o}(\Delta t^2) \end{aligned} \tag{7}$$

by equation (4). Now one can estimate, with $t = n\Delta t + t_0, t_i = \mathrm{i}\Delta t + t_0$,

$$\begin{aligned} |P_\perp(t, t_0; |\psi\rangle) - \|\mathbb{P}_\perp|\psi\rangle\|^2| &\leqslant \sum_{i=1}^{n-1} |P_\perp(t_{i+1}, t_0; |\psi\rangle) - P_\perp(t_i, t_0; |\psi\rangle)| \\ &\quad + \left|P_\perp(t_1, t_0; |\psi\rangle) - \|\mathbb{P}_\perp|\psi\rangle\|^2\right|. \end{aligned} \tag{8}$$

The sum is bounded by $(n-1)\Delta t^2$ constant $+ (n-1)\mathrm{o}(\Delta t^2)$, and for $\Delta t \to 0$ this vanishes, as does the last term on the r.h.s. For $H = H(t)$ time dependent, the same argument goes through with minor modifications.

For $|a\rangle$ in the domain of H and initial state $|\psi\rangle$, this simple argument shows that for rapidly repeated ideal measurement of $\mathbb{P}_a = |a\rangle\langle a|$ the results freeze, for $\Delta t \to 0$, to $|a\rangle$ with probability $|\langle a|\psi\rangle|^2$ and to $\mathbb{P}_\perp|\psi\rangle$ with the complementary probability. In particular, if $|\psi\rangle = |a\rangle$, one stays in $|a\rangle$ for $\Delta t \to 0$.

2.1. Mean length of periods

For a single system one has as results of the measurement alternating random sequences of a's and $\perp$'s ($\equiv$ not a) of the form

$$\ldots \perp aa \ldots a \perp\perp \ldots \perp a \ldots. \tag{9}$$

The length of an a sequence is defined as $\Delta t\ \times$ number of a's. Similarly for $\perp$. We assume that $|a\rangle$ is not an eigenvector of H, since otherwise all measurements would give the same result, either all a or all not a ($\perp$). The initial state for an a sequence is $|a\rangle$ and for an $\perp$ sequence it is

$$|\phi_\perp\rangle \equiv \mathbb{P}_\perp U(\Delta t, 0)|a\rangle/\|\cdot\| \tag{10}$$

except at the beginning when it is $|\psi\rangle$.

Starting with an a the probability to have exactly n a's in a row, $n \geqslant 1$, but not more, is by equation (1) (with $t_0 = 0$)

$$\begin{aligned} p_{a;n} &= \|\mathbb{P}_\perp U(\Delta t, 0)\psi_a(t_{n-1}, 0; |a\rangle)\|^2 \\ &= P_a(t_{n-1}, 0; |a\rangle) - P_a(t_n, 0; |a\rangle) \end{aligned} \tag{11}$$

and analogously

$$p_{\perp;n} = P_\perp(t_{n-1}, 0; |\phi_\perp\rangle) - P_\perp(t_n, 0; |\phi_\perp\rangle). \tag{12}$$

The mean duration T_a and $T_\perp$ of these sequences for a single system is then, in obvious notation,

$$\begin{aligned} T_{a,\perp} &= \sum_{n=1}^{\infty} n\Delta t[P_{a,\perp}(t_{n-1}) - P_{a,\perp}(t_n)] \\ &= \sum_{n=0}^{\infty} \Delta t P_{a,\perp}(t_n). \end{aligned} \tag{13}$$

From equation (2) one obtains the exact result

$$\begin{aligned} T_a &= \Delta t \sum_{n=0}^{\infty} |\langle a|U(\Delta t, 0)|a\rangle|^{2n} \\ &= \frac{\Delta t}{1 - |\langle a|U(\Delta t, 0)|a\rangle|^2}. \end{aligned} \tag{14}$$

With equation (4) one obtains

$$T_a = \frac{1}{\Delta t}\left\{\frac{\hbar^2}{\langle a|H^2|a\rangle - \langle a|H|a\rangle^2} + \mathrm{o}(\Delta t^2)/\Delta t^2\right\}. \tag{15}$$

The second term in the brackets becomes negligible for small Δt, and T_a diverges for $\Delta t \to 0$. If $|a\rangle$ is in the domain of H^2 then one can replace $\mathrm{o}(\Delta t^n)$ by $\mathrm{O}(\Delta t^{n+1})$ where the latter denotes terms of order at least Δt^{n+1}.

To obtain an explicit expression for $T_\perp$ we assume for simplicity that the Hilbert space is finite-dimensional (or that H is bounded). Then one has

$$\begin{aligned} \mathbb{P}_\perp U(\Delta t, 0)\mathbb{P}_\perp &= \mathbb{P}_\perp[\mathbb{1} - \mathrm{i}\Delta t H/\hbar - \tfrac{1}{2}\Delta t^2 H^2/\hbar^2 + \mathrm{O}(\Delta t^3)]\mathbb{P}_\perp \\ &= \mathbb{P}_\perp \mathrm{e}^{-\mathrm{i}\Delta t\mathbb{P}_\perp H\mathbb{P}_\perp/\hbar - \frac{1}{2}\Delta t^2[\mathbb{P}_\perp H^2\mathbb{P}_\perp - (\mathbb{P}_\perp H\mathbb{P}_\perp)^2]/\hbar^2}\mathbb{P}_\perp(1 + \mathrm{O}(\Delta t^3)). \end{aligned} \tag{16}$$

Then, by equation (3)

$$P_\perp(t_n, 0; |\psi_\perp\rangle) = \langle\psi_\perp|\mathbb{P}_\perp \mathrm{e}^{-n\Delta t^2[\mathbb{P}_\perp H^2\mathbb{P}_\perp - (\mathbb{P}_\perp H\mathbb{P}_\perp)^2]/\hbar^2}\mathbb{P}_\perp|\psi_\perp\rangle(1 + \mathrm{O}(\Delta t^3)). \tag{17}$$

From this and from equation (13) one now obtains

$$T_\perp = \frac{1}{\Delta t}\langle\phi_\perp|\frac{\hbar^2}{\mathbb{P}_\perp H^2\mathbb{P}_\perp - (\mathbb{P}_\perp H\mathbb{P}_\perp)^2}|\phi_\perp\rangle + \mathrm{O}(\Delta t). \tag{18}$$

We note that if $|a\rangle$ is an eigenvector of H then the denominators in equations (14) and (18) vanish.

Example. We consider a single system with two stable levels one and two. The system is driven in resonance by a classical electromagnetic wave, e.g. in the RF range. In the interaction picture and with the usual rotating-wave approximation the Hamiltonian is given by

$$H = \tfrac{\hbar}{2}\Omega_2\{|1\rangle\langle 2| + |2\rangle\langle 1|\} \tag{19}$$

where Ω_2, the so-called Rabi frequency, is proportional to the amplitude of the driving field [17, 18]. The time-development operator is easily calculated as

$$U(t, t_0) = \cos \tfrac{1}{2}\Omega_2(t - t_0) - \mathrm{i} \sin \tfrac{1}{2}\Omega_2(t - t_0)\{|1\rangle\langle 2| + |2\rangle\langle 1|\}. \tag{20}$$

From this one finds the transition probabilities

$$|\langle 2|U(t, 0)|1\rangle|^2 = |\langle 1|U(t, 0)|2\rangle|^2 = \sin^2 \tfrac{1}{2}\Omega_2 t. \tag{21}$$

For small t this is quadratic in t. If one now determines by repeated ideal measurements, at times Δt apart, whether one finds the system in state $|1\rangle$ or $|2\rangle$ one obtains a random sequence of the form

$$\ldots 21 \ldots 12 \ldots 21 \ldots \tag{22}$$

similar to (9). The mean duration T_1 and T_2 of the subsequences of 1's and 2's is given by equation (14) with $|a\rangle$ replaced by $|1\rangle$ and $|2\rangle$, respectively, and one obtains with equation (20)

$$T_1 = T_2 = \frac{\Delta t}{\sin^2 \frac{1}{2}\Omega_2 \Delta t} = \frac{4}{\Omega_2^2 \Delta t} + \mathrm{O}(\Delta t). \tag{23}$$

Note that $T_1 = T_2$ holds quite generally for a two-level system, as easily seen from equation (14).

3. Realistic case: light and dark periods

We now consider a single three-level V system as in figure 1 and assume the 1–2 transition to be driven in resonance by classical electromagnetic RF radiation with Rabi frequency Ω_2 and Hamiltonian as in equation (19).

We suppose that repeated measurements of level one are performed. Following [7, 6] we assume that each measurement consists of a short laser (probe) pulse driving the 1–3 transition. When resonance fluorescence occurs then after the last photon emission at the end of a probe pulse the system is in $|1\rangle$, and when no resonance fluorescence occurs then the system was taken by [7, 6] to be in $|2\rangle$.

Experimentally one will then expect the following striking phenomenon. One will see periods of fluorescence bursts alternating with dark periods, as in figure 2. The mean duration of these light and dark periods should be given by $T_{1,2}$ of equation (23), at least approximately,

$$T_L \cong \frac{4}{\Omega_2^2 \Delta t} \qquad T_D \cong \frac{4}{\Omega_2^2 \Delta t}. \tag{24}$$

These periods should become longer and longer with decreasing time Δt between the probe pulses.

In how far the above probe pulses do indeed lead to measurements of levels 1 and 2 and to state reduction has recently been discussed by us in [10–12] by means of the quantum jump approach [13]. With regards to reduction, it was shown that at the end of a probe pulse and a short transitory time the state of the system is given either by a density matrix extremely close, but not identical to $|1\rangle\langle 1|$ if the system has emitted photons, or by a density matrix very close to $|2\rangle\langle 2|$ if no photons were emitted. After the last photon emission during a probe pulse the system is indeed in its ground state, but then it may acquire a small $|2\rangle$ component until the end of the probe pulse; its $|3\rangle$ component will decay during a short transitory time after the pulse. When no photons are emitted the finite duration of the probe

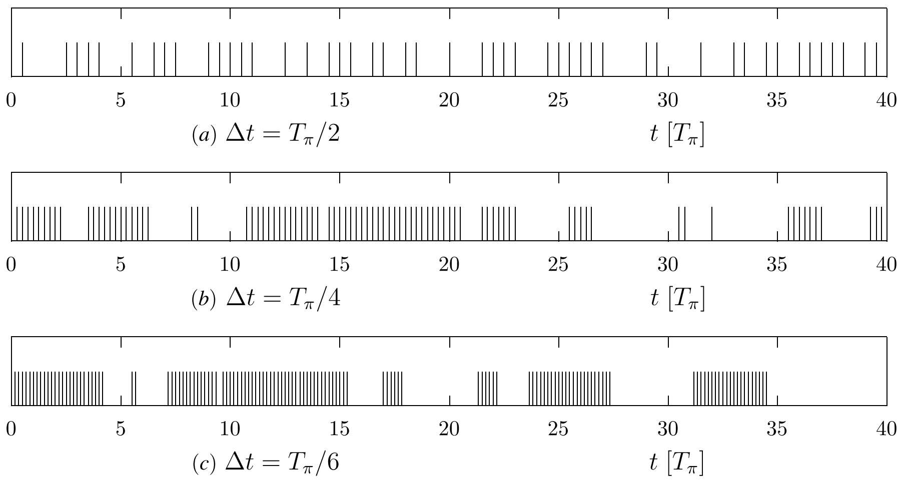

Figure 2. Figure 2. Stochastic alternating light and dark periods. The lines mark times when the atom is found in state $|1\rangle$ and emits a burst of light. $T_\pi = \pi/\Omega_2$ is the length of a π pulse.

pulse is responsible for a small $|1\rangle$ component. Hence, there will be small deviations from ideal measurements, which will lead to small corrections to the above results.

For a probe pulse to constitute an effective measurement its duration $\Delta\tau_{\rm p}$ has to satisfy [10]

$$\Delta\tau_{\rm p} \gg \max\{A_3^{-1}, A_3/\Omega_3^2\}. \tag{25}$$

In addition to this one needs

$$\epsilon_{\rm p} \equiv \frac{\Omega_2 A_3}{\Omega_3^2} \ll 1 \qquad \epsilon_{\rm R} \equiv \frac{\Omega_2}{\Omega_3} \ll 1 \qquad \epsilon_{\rm A} \equiv \frac{\Omega_2}{A_3} \ll 1. \tag{26}$$

If the time Δt between two probe pulses satisfies

$$\Delta t \gg A_3^{-1} \qquad \text{and} \qquad (\Omega_2 \Delta t)^2 \gg \epsilon \tag{27}$$

one can directly employ the results of [11]. The first of these conditions ensures that the $|3\rangle$ component has vanished before the next pulse, the second that there are only two possible atomic states at the end of a pulse. In case of no emission the pulse effectively projects the system onto

$$\tilde{\rho}_{\rm P}^0 = \begin{pmatrix} 0 & -{\rm i}\epsilon_{\rm p} \\ {\rm i}\epsilon_{\rm p} & 1 \end{pmatrix} + {\rm O}(\epsilon^2) \tag{28}$$

in the $|1\rangle - |2\rangle$ subspace, and in case of photon emission onto

$$\tilde{\rho}_{\rm P}^> = \frac{1}{A_3^2 + 2\Omega_3^2 + \epsilon_{\rm p}\Omega_2\Delta\tau_{\rm p}A_3^2} \times \begin{pmatrix} A_3^2 + 2\Omega_3^2 & {\rm i}\epsilon_{\rm p}A_3^2 - \frac{\rm i}{2}\epsilon_{\rm A}\Omega_3^2 \\ -{\rm i}\epsilon_{\rm p}A_3^2 + \frac{\rm i}{2}\epsilon_{\rm A}\Omega_3^2 & \epsilon_{\rm p}\Omega_2\Delta\tau_{\rm p}A_3^2 \end{pmatrix} + {\rm O}(\epsilon^2). \tag{29}$$

For arbitrary initial density matrix ρ the probability for no photon emission during a probe pulse is

$$P_0(\Delta\tau_{\rm p}; \rho) = \rho_{22} - \epsilon_{\rm p}\Omega_2\Delta\tau_{\rm p}\rho_{22} + 2\epsilon_{\rm p}\,{\rm Im}\,\rho_{12} - 2\epsilon_{\rm R}\,{\rm Re}\,\rho_{23} + {\rm O}(\epsilon^2). \tag{30}$$

Now let p be the (conditional) probability to have *no* fluorescence during a pulse under the condition that there *had* been fluorescence during the preceding pulse. By q we denote the probability to have *no* fluorescence during a pulse under the condition that there had been *no* fluorescence during the preceding pulse. In short, p and q are transition probabilities,

$$p : \text{yes} \rightarrow \text{no} \qquad q : \text{no} \rightarrow \text{no}. \tag{31}$$

These are the same probabilities as for the transitions from $\tilde{\rho}_{\rm P}^{>}$ after a pulse to $\tilde{\rho}_{\rm P}^{0}$ after the next pulse and from $\tilde{\rho}_{\rm P}^{0}$ to $\tilde{\rho}_{\rm P}^{0}$, respectively. With

$$c \equiv \cos \Omega_2 \Delta t \qquad s \equiv \sin \Omega_2 \Delta t \tag{32}$$

one has [11]

$$p = \frac{1}{2}(1-c) + \epsilon_{\rm p}\left\{2s\frac{A_3^2+\Omega_3^2}{A_3^2+2\Omega_3^2} + \frac{1}{2}\Omega_2\Delta\tau_{\rm p}c\frac{3A_3^2+2\Omega_3^2}{A_3^2+2\Omega_3^2} - \frac{1}{2}\Omega_2\Delta\tau_{\rm p}\right\} - \frac{1}{2}\epsilon_{\rm A}s\frac{\Omega_3^2}{A_3^2+2\Omega_3^2} + {\rm O}(\epsilon^2) \tag{33}$$

$$q = \tfrac{1}{2}(1+c) - \epsilon_{\rm p}\{2s + \tfrac{1}{2}\Omega_2\Delta\tau_{\rm p}(1+c)\} + {\rm O}(\epsilon^2). \tag{34}$$

It should be noted that for small Δt

$$p = \tfrac{1}{4}(\Omega_2\Delta t)^2 + {\rm O}(\epsilon) \tag{35}$$

$$q = 1 - p + {\rm O}(\epsilon) \tag{36}$$

and that $q \neq 1 - p$ to first order in ϵ.

The probability for a period of exactly n consecutive probe pulses *with* fluorescence among all such light periods is $(1-p)^{n-1}p$. The mean duration T_L of light periods is then

$$T_L = \sum_{n=1}^{\infty}(\Delta\tau_{\rm p} + \Delta t)n(1-p)^{n-1}p \tag{37}$$

which gives

$$T_L = \frac{\Delta\tau_{\rm p} + \Delta t}{p}. \tag{38}$$

Similarly one finds for the dark periods

$$T_D = \frac{\Delta\tau_{\rm p} + \Delta t}{1-q}. \tag{39}$$

Since $1-q$ is close, but not equal, to p one has $T_L \approx T_D$ but no longer equality. For the parameters of [6] the difference is very small.

Inserting the approximate values of p and q from equations (35) and (36) one obtains

$$T_L \approx T_D \approx \frac{\Delta\tau_{\rm p} + \Delta t}{\Delta t}\frac{4}{\Omega_2^2\Delta t}. \tag{40}$$

If the duration $\Delta\tau_{\rm p}$ of the probe pulse is much smaller than the time Δt between the pulses this agrees extremely well with the result for ideal measurements obtained by the projection postulate in equations (23) and (24) above.

It is not possible to take the limit $\Delta t \rightarrow 0$ in equation (40) since for the above derivation to be valid Δt has to satisfy $\Delta t \gg A_3^{-1}$. This limit will be studied in the next section, and we will show that T_L and T_D do not grow indefinitely.

4. The limit of vanishing distance between probe pulses: $\Delta t \to 0$

To perform the limit $\Delta t \to 0$ some extra steps are needed. For small Δt the population of level three does not vanish completely before the beginning of the next probe pulse. Therefore, in case of fluorescence, one has no longer a good reduction to $|1\rangle\langle 1|$ and the pulse cannot be regarded as affecting a measurement of levels one and two. In this case the treatment of the last section has to be made more precise by incorporating the possibly only partial decay of level three.

Right at the end of a probe pulse—without transient decay time—the system is, as shown in [11], either in

$$\tilde{\rho}^0 = \begin{pmatrix} 0 & -\mathrm{i}\epsilon_\mathrm{p} & 0 \\ \mathrm{i}\epsilon_\mathrm{p} & 1 & -\epsilon_\mathrm{R} \\ 0 & -\epsilon_\mathrm{R} & 0 \end{pmatrix} + \mathrm{O}(\epsilon^2) \tag{41}$$

in case of no photon emission, or in

$$\tilde{\rho}^> = \frac{1}{A_3^2 + 2\Omega_3^2 + \epsilon_\mathrm{p} A_3^2 \Omega_2 \Delta\tau_\mathrm{p}} \begin{pmatrix} A_3^2 + \Omega_3^2 & \mathrm{i}\epsilon_\mathrm{p} A_3^2 & \mathrm{i}A_3\Omega_3 \\ -\mathrm{i}\epsilon_\mathrm{p} A_3^2 & \epsilon_\mathrm{p} A_3^2 \Omega_2 \Delta\tau_\mathrm{p} & \epsilon_\mathrm{R}(A_3^2+\Omega_3^2) \\ -\mathrm{i}A_3\Omega_3 & \epsilon_\mathrm{R}(A_3^2+\Omega_3^2) & \Omega_3^2 \end{pmatrix} + \mathrm{O}(\epsilon^2) \tag{42}$$

in case of fluorescence, except possibly for the *first* pulse of a light period. If the second condition in equation (27) is not satisfied by Δt then the state at the beginning of the first pulse in a light period is very close to ρ^0, and therefore the state $\tilde{\rho}^>$ after the first pulse has to be calculated with initial state of the form $\rho^0 + \mathrm{O}(\epsilon)$. For such a state, however, one has $1 - P_0 = \mathrm{O}(\epsilon)$, by equation (30), and then $\mathrm{O}(\epsilon^2)$ is replaced by $\mathrm{O}(\epsilon)$ in equation (42) for small Δt. Thus, if the second condition in equation (27) does not hold the first pulse in a light period has, in principle, to be treated differently from the rest.

The transition probabilities from equation (31) are now denoted by $\tilde{p}$ and $\tilde{q}$ and are given by

$$\tilde{p} = p - 2\epsilon_\mathrm{R} s \frac{\Omega_3 A_3}{A_3^2 + 2\Omega_3^2} \mathrm{e}^{-\frac{1}{2}A_3\Delta t} + \mathrm{O}(\epsilon^2) \tag{43}$$

$$\tilde{q} = q + \mathrm{O}(\epsilon^2) \tag{44}$$

with p and q as in equations (33) and (34) and Δt arbitrary. However, for the first pulse in a light period $\tilde{p}$ is replaced by $\tilde{p} + \mathrm{O}(\epsilon)$. One sees that, for $\Delta t \gg A_3^{-1}$, $\tilde{p}$ goes over into p. Equation (37) is replaced by

$$T_L = (\Delta\tau_\mathrm{p} + \Delta t)(\tilde{p} + \mathrm{O}(\epsilon)) + \sum_{n=2}^{\infty} (\Delta\tau_\mathrm{p} + \Delta t) n (1 - \tilde{p} + \mathrm{O}(\epsilon))(1-\tilde{p})^{n-2}\tilde{p} \tag{45}$$

which gives

$$T_L = \frac{\Delta\tau_\mathrm{p} + \Delta t}{\tilde{p}} \tag{46}$$

up to terms of relative order ϵ. For T_D one obtains now

$$T_D = \frac{\Delta\tau_\mathrm{p} + \Delta t}{1 - \tilde{q}}. \tag{47}$$

Now one performs the limit $\Delta t \to 0$ and obtains

$$\begin{aligned} \lim_{\Delta t \to 0} \tilde{p} &= \epsilon_\mathrm{p} \Omega_2 \Delta\tau_\mathrm{p} \frac{A_3^2}{A_3^2 + 2\Omega_3^2} + \mathrm{O}(\epsilon^2) \\ \lim_{\Delta t \to 0} \tilde{q} &= 1 - \epsilon_\mathrm{p}\Omega_2\Delta\tau_\mathrm{p} + \mathrm{O}(\epsilon^2). \end{aligned} \tag{48}$$

Inserting this into the expressions for T_L and T_D gives, with $\epsilon_{\rm p} = \Omega_2 A_3/\Omega_3^2$,

$$\begin{aligned}\lim_{\Delta t\to 0} T_L &= \frac{A_3^2 + 2\Omega_3^2}{\Omega_2^2 A_3^3}\Omega_3^2 \\ \lim_{\Delta t\to 0} T_D &= \frac{\Omega_3^2}{\Omega_2^2 A_3}\end{aligned} \tag{49}$$

up to terms of relative order $\epsilon/\Omega_2\Delta\tau_{\rm p}$.

First of all, the limits are finite, as physically expected. Furthermore, in the limit $\Delta t \to 0$ both driving fields are continuously on and in this case the existence of macroscopic light and dark periods is well known under the name 'electron shelving' [14]. The mean duration of these periods has been calculated [19] and the result is the same as in equation (49). Thus, the continuously driven case is recovered in the limit $\Delta t \to 0$.

5. Conclusion

When applied to an ensemble of systems the QZE predicts a slow-down in the time-development of the density matrix $\rho(t)$ under repeated ideal measurements. An experiment to test this was performed by Itano *et al* [6] in which repeated state measurements were carried out on a system with two stable levels $|1\rangle$ and $|2\rangle$. The measurements were implemented by short laser pulses driving the transition from the ground state $|1\rangle$ to an auxiliary rapidly decaying level $|3\rangle$. Occurrence or absence of fluorescence means a system is in $|1\rangle$ or $|2\rangle$, respectively. The experimental results indeed showed a slow-down of the time-development of $\rho(t)$ in good agreement with the QZE. Subsequently it was pointed out [9] that this behaviour could be understood without recourse to any measurement theory. Indeed, one can simply consider the probe laser as part of the dynamics and incorporate it in the Hamiltonian or in the Bloch equations for $\rho(t)$, never speaking of measurements. Using the quantum jump approach [13] (or quantum trajectories) it is possible to understand why the dynamics is so well described by notion of measurements and by the projection postulate [10, 11].

Instead of an ensemble of atoms we have considered a *single* three-level V system, with the same weak field driving the $|1\rangle - |2\rangle$ transition and laser pulses driving the $|1\rangle$–$|3\rangle$ transition as before. Taking the measurement point of view, the projection postulate gives a quick and intuitive understanding what to expect, namely a stochastic sequence of fluorescence bursts (light periods) and dark periods, as in figure 2. Their durations should increase with decreasing distance between the laser pulses.

Taking the dynamical point of view, Bloch equations are not so convenient, but the quantum jump approach is particularly well adapted to single systems. Using this approach we have shown in this paper why, and for which parameter values, the simple projection postulate prescription gives so highly accurate results. We have not only calculated corrections to the projectile-postulate results, but we have also shown that if the time Δt between the laser pulses becomes too short then the projection postulate can no longer be applied. The quantum jump approach, however, can also handle the limit $\Delta t \to 0$ and yields convergence to the well known light and dark periods of the continuously driven system [14, 19]. These dark periods are also called electron shelving since during this time the system is predominantly in $|2\rangle$. For an ensemble of many atoms different light and dark periods will overlap, and as a result only a lower intensity of fluorescence will be seen.

If the duration of a probe pulse becomes too short the measurement picture is also not applicable, but the quantum jump approach still is. In this case a numerical simulation is easiest.

In summary, we have demonstrated the usefulness of the projection postulate for the stochastic behaviour of a single system. Our dynamical analysis also clearly shows that the projection postulate is an idealization, sometimes even an over-idealization, and that in a more precise treatment corrections arise. Experimentally, it should be possible to check our results for a single ion or atom in a trap.

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